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On the Solutions of Linear Odd-Order Heat-Type Equations with Random Initial Conditions[1]

L. Beghin,[2] Yu. Kozachenko,[3] E. Orsingher[2] and L. Sakhno[3]

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In this paper odd-order heat-type equations with different random initial conditions are examined. In particular, we give rigorous conditions for the existence of the solutions in the case where the initial condition is represented by a strictly φ–subGaussian harmonizable process $\eta = \eta(x)$. Also the case where η is represented by a stochastic integral with respect to a process with independent increment is studied.

KEY WORDS: Higher-order heat-type equations, Harmonizable processes, φ–subGaussian processes

1. INTRODUCTION

Odd-order parabolic and hyperbolic partial differential equations emerge in several applied fields.

In the book by Gardiner (1985) at page 295 a trimolecular (unstable) chemical reaction is studied and the corresponding third-order Fokker-Planck equation (7.7.114) is derived. There are various derivations of this third-order p.d.e. (based on the Poisson representation, see Ref. 7, page 300) and, in some contexts, in order to study the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = \frac{1}{\gamma}\left\{\left(-\frac{\partial}{\partial x} + 2\frac{\partial^2}{\partial x^2} - \frac{\partial^3}{\partial x^3}\right)(a-x)x^2 p\right\} \tag{1.1}$$

[1] Partially supported by the NATO grant PST.CLG.980408.
[2] Dipartimento di Statistica, Probabilità e Statistiche Applicate, University of Rome “La Sapienza”, p.le Aldo Moro 5, 00185, Rome, Italy; e-mail: enzo.orsingher@uniroma1.it
[3] Department of Mechanics and Mathematics, Kyiv National Taras Shevchenko University, Kyiv, 01033, Ukraine.

it is necessary to analyze stochastic differential equations of the form

$$dy = a\,dt + b dW + c dV,$$

where W and V are independent processes. The component V is a third-order noise whose signed distribution $p = p(v, t)$ is governed by equations of the form

$$\frac{\partial p}{\partial t} = -\frac{1}{6}\frac{\partial^3 p}{\partial v^3}. \tag{1.2}$$

Equation (1.2) also emerges as a linear approximation of the Korteweg-de Vries equation (see, e.g., Ref. 2 and Sec. 2 below).

The fundamental solutions of these third-order equations are signed and, based on them, the so-called pseudoprocesses have been constructed and their properties studied. Moreover some of the related relevant functionals (sojourn time and maximum) have been investigated by applying a generalization of the Feynman-Kac functional in Ref. 29. In Ref. 1 the case where the pseudoprocess is constrained to be zero at the end of the time interval is considered; the distribution of the maximum is then obtained under these circumstances. In the unconditional case, the joint distribution of the maximum and of the process for this higher-order diffusion is presented in Ref. 4.

We mention also another source of third-order equations, which is represented by the random motions at finite velocity on the line or on the plane with three possible velocities or directions. Since the order of the equations governing this type of finite-velocity motions equals the number of possible directions (or velocities on the line) we can deal with equations of any order (including the odd-order ones examined in this paper).

Odd-order heat-type equations of the form

$$\frac{\partial u}{\partial t} = c_n \frac{\partial^{2n+1} u}{\partial x^{2n+1}}, \qquad n = 1, 2, \ldots \tag{1.3}$$

(where $c_n = \pm 1$), subject to the initial condition $u(x, 0) = \delta(x)$, have also been examined by many authors: in Ref. 9 the Laplace transforms of the sojourn times have been obtained while their inverse, and thus the explicit distributions, have been derived in Ref. 21.

In Ref. 3 the analysis of the local time in zero of the pseudoprocesses related to (1.3) is performed and the connection of its distribution with a fractional diffusion equation is established and discussed.

While in all the investigations mentioned above the key tool is the Feynman-Kac functional, the approach of Ref. 22 is somewhat different and consists in some approximation of the underlying pseudoprocesses by means of generalized random walks and the application of a generalization of the Spitzer identity.

The idea of studying equations of the form (1.1) subject to random initial conditions (represented by stationary processes) is presented in Ref. 2. In the spirit of the last work, we analyze here more general odd-order equations of the following form

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{N} a_k \frac{\partial^{2k+1} u}{\partial x^{2k+1}}, \qquad N = 1, 2, \ldots, \tag{1.4}$$

subject to the random condition

$$u(0, x) = \eta(x), \tag{1.5}$$

where $\eta(x) = \int_R e^{iux} dy(u)$ and y is a complex-valued process. We remark that, in the special case where η is a stationary process, y is a white noise. We present the exact expression for the solution to the problem (1.4)–(1.5) and formulate rigorous conditions on the initial data which guarantee that the process representing the solution satisfies the equation with probability one (Sec. 4).

We concentrate our attention, in particular, on the case where the initial condition is represented by a strictly φ–subGaussian harmonizable process. The general conditions of Sec. 4 are reduced to a more convenient and tractable form (see our main result in Sec. 5). We consider also the problem where the initial data is represented by a stochastic integral with respect to a process with independent increments (Sec. 6). Consult, on this point, for stable processes, Ref. 28 and, for infinitely divisible processes, Ref. 30.

We note that random processes relevant for applications (as numerous recent studies confirm) display a non-Gaussian behavior, possess heavy tails and have non-symmetric densities. For example, in the case of the usual heat equation (and also for the third-order heat-type equations appearing in trimolecular chemical reactions) the non-homogeneous structure of the material makes non-symmetric distributions for the initial conditions more plausible. However, some of these processes can be considered as φ–subGaussian because they possess the corresponding properties. φ–subGaussian random variables and processes, which are generalizations of sub–Gaussian and Gaussian random variables and processes, were introduced in the papers[16,18]. The theory of φ–subGaussian random variables and processes is presented in the book[5]. In Ref. 8 a more general definition of φ–subGaussian random variables is introduced.

In order to make the paper self-contained, a certain digression on sub-Gaussian and φ–subGaussian processes is presented in the Appendix together with some auxiliary results needed to treat the case of initial condition represented by stochastic integrals with respect to processes with independent increments. Note that the case of Gaussian initial conditions is also covered by our study, as a special case.

We would like to stress the importance of the approach developed in the paper: we provide conditions on random initial data, which guarantee that the solution presented here satisfies the equation with probability one. This permits us to relate the solution to the equations considered here with the original physical problem in a rigorous way.

2. MOTIVATION FOR OUR STUDY

Equation (1.4) pertains to the class of linear evolution equations. Moreover, the coefficients of even order of the polynomial $P(\frac{\partial}{\partial x})$ in the right hand side of (1.4) are zero, and under the assumption that all the odd order coefficients a_k of $P(\frac{\partial}{\partial x})$ are real, we have a linear wave equation which admits plane wave solutions. The related initial value problems are interesting for their own sake, but also because they can serve as a tool for studying nonlinear equations, as we will show below.

Equations of the form (1.4) are also called dispersive equations. In the simplest case, for $N = 1$, (and for k starting from 0) we have

$$\frac{\partial u}{\partial t} = a_0 \frac{\partial u}{\partial x} + a_1 \frac{\partial^3 u}{\partial x^3} \tag{2.1}$$

(also called the "weak dispersion" wave equation), which can be transformed into the paradigmatic form of a third-order heat-type equation

$$\frac{\partial u}{\partial t} = b \frac{\partial^3 u}{\partial x^3}. \tag{2.2}$$

The above equations (known also as Airy equations) represent the linearized version of the celebrated Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0. \tag{2.3}$$

The KdV equation is used to model the propagation of small amplitude unidirectional irrotational long waves on the surface of an inviscid fluid in a flat channel (see, e.g., Ref. 33). Besides the wave propagation in water and fluids, the KdV equation can be useful in the description of certain waves in plasma and in a variety of other media. Under some specific assumptions or in some limiting situations the study can be reduced to the waves which follow either the linearized KdV Eqs. (2.1) or (2.2). A number of analytical results and computer simulations for KdV equation and its generalizations (with the nonlinear term of a more general form) show that its solitary wave solutions evolve accompanied by small dispersive waves which (being considered in a certain moving frame) develop approximately according to Eq. (2.1).

Furthermore, when we are interested in solving the initial value problem for the KdV equation, it is very instructive to study separately its linear and nonlinear counterparts and this is especially useful if we think of computer simulations of solutions. Namely, the Cauchy problem for the KdV equation can be solved numerically by using the so-called split-stepping method to combine the solutions for $\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial t} + \beta \frac{\partial^3 u}{\partial x^3} = 0$ (see, e.g., Ref. 31). This justifies the study of the Cauchy problem for the linear counterpart of the KdV equation.

It should be noted that the KdV equation is obtained, at a certain level of approximation, when higher-order dispersive effects have been neglected. However, in many cases, physical reality requires more accuracy. Under certain circumstances it may happen that higher-order dispersive terms (e.g., the fifth-order term) have a significant role in the wave propagation process. This is confirmed by numerous studies devoted to the investigation of generalized KdV-type equations, e.g. the equations with the same nonlinear term and the fifth-order dispersive term instead of (or in addition to) the third-order one. We mention here some of such contributions. Numerical studies for the KdV-type equations containing only the fifth-order dispersive term were carried out in Refs. 27, 34 and others. The KdV-type equation with both the third- and fifth-order dispersive term was first proposed by Kakutani and Ono[(13)] in the study of magneto-acoustic waves in cold collision-free plasma. Similar equations were studied in Ref. 11 as a model for capillary-gravity waves. Kawahara[(14)] revealed different structure of wave solutions to such equations depending on which is the dominating term (either the third-order or the fifth-order one). Nagashima[(26)] studied the behavior of systems governed by such equations and showed that the type of motion in the system (regular or chaotic) is determined by the initial condition and the coefficients of the third-order and fifth-order derivatives. We note also that, in some recent studies, wave propagation in microstructured media is modelled by even more complicated KdV-type equations, with higher-order nonlinearity and higher order dispersive terms, taken into account in order to compensate this nonlinearity.

Dealing with the initial value problem for the generalized KdV-type equations mentioned above and the corresponding numerical computations (e.g. by means of split-stepping method), it will be useful again to study their linear counterparts, represented by equations of the form (1.4).

The above arguments justify the study of equations of the form (1.4). Moreover, in the present paper we consider the Cauchy problem with a random initial condition, which is relevant in many practical situations. As a matter of fact, in nature, waves exhibit random character. There exist numerous investigations of the propagation of waves in random media which exhibit the KdV-type behavior. Some papers related to these investigations are devoted to the propagation of an initially deterministic wave controlled by a randomly perturbed KdV equation (see, e.g., Ref. 32 for the case of equation with a white noise, Ref. 12 for the case of a noise with long-range correlation, among many others). Another adequate statistical description of the random character of wave propagation under different circumstances is provided by assuming random initial conditions for the non-random equation (here we mention, e.g., Refs. 2 and 15 for KdV and Airy equations). Note that in many papers of the physical literature it is usual to characterize the time evolution of small amplitude surface gravity waves under Gaussian assumptions. However, the behavior of these waves may become highly non-Gaussian, as many empirical studies demonstrate. This suggests to choose non-Gaussian initial conditions for our investigation, namely, a φ–subGaussian one.

3. HARMONIZABLE RANDOM PROCESSES

We now present the definitions of integrals in the mean square sense and also of the harmonizable random processes (see, for example, Ref. 24).

Let $y = \{y(t), t \in I\}$ be a complex-valued, centered random process of second order (that is $E|y(t)|^2 < \infty, t \in I$), $I = [a, b]$ a finite or infinite interval and $\Gamma_y(t, s) = Ey(t)\overline{y(s)}$ the covariance function of $y(t)$.

Definition 3.1. [[24]] Let D and D' be the following partitions of the interval $[a, b]$:

$$D = \{t_j, j = 1, \ldots, n+1 : a = t_1 < t_2 < \ldots < t_{n+1} = b\};$$
$$D' = \{t'_j, j = 1, \ldots, m+1 : a = t'_1 < t'_2 < \ldots < t'_{m+1} = b\}.$$

Let also

$$\Delta\Delta'\Gamma_y(t_k, t'_k) = \Gamma_y(t_{k+1}, t'_{k+1}) - \Gamma_y(t_{k+1}, t'_k) - \Gamma_y(t_k, t'_{k+1}) + \Gamma_y(t_k, t'_k).$$

The covariance function $\Gamma_y(t, s)$ has finite variation on the finite interval $I = [a, b]$ if there exists a number $0 < C_I < \infty$ such that, for all D and D', the following inequality holds $\sum_{t\in D}\sum_{t\in D'} |\Delta\Delta'\Gamma_y(t, t')| < C_I$.

The covariance function $\Gamma_y(t, s)$ has finite variation on the infinite interval I if there exists a number $C < \infty$ such that $C_{I'} < C$ for all finite I' such that $I' \subset I$.

Definition 3.2. [[24]] Let $f = \{f(t), t \in I\}$ be a measurable function (where $I = [a, b]$ is a finite interval), $y = \{y(t), t \in I\}$ a centered second-order random process and $\Gamma_y(t, s) = Ey(t)\overline{y(s)}$ the covariance function of y. The integral $\int_I f(t)\, dy(t)$ is defined as the mean square limit of the Riemann sums $\sum_k f(t'_k)(y(t_{k+1}) - y(t_k))$, $t_k \le t'_k \le t_{k+1}$. The integral $\int_R f(t)\, dy(t)$ is defined as the mean square limit of the integrals $\int_{-a}^{b} f(t)\, dy(t)$ as $a \to \infty, b \to \infty$. The integral $\int_I f(t)\, dy(t)$ exists iff the integral $\int_I \int_I f(t)f(s) d\Gamma_y(t, s)$ exists. The reader can consult[25] for extensions of the Riemann-Stieltjes integrals.

Definition 3.3. [[24]] The second-order random function $X = \{X(t), t \in R\}$ is called harmonizable if there exists a second-order random function $y = \{y(t), t \in R\}$ such that the covariance $\Gamma_y(t, s) = Ey(t)\overline{y(s)}$ has finite variation and $X(t) = \int_R e^{itu} dy(u)$.

Theorem 3.1. [[24]] *The second-order random function $X = \{X(t), t \in R\}$ is harmonizable iff there exists a covariance function $\Gamma_y(t, s)$ with finite variation such that $\Gamma_x(u, v) = \int_R \int_R e^{i(tu-t'v)} d\Gamma_y(t, t')$.*

4. A GENERAL THEOREM ON THE SOLUTION OF ODD-ORDER HEAT-TYPE EQUATIONS

Let us consider the linear equation

$$\sum_{k=1}^{N} a_k \frac{\partial^{2k+1} u(t, x)}{\partial x^{2k+1}} = \frac{\partial u(t, x)}{\partial t}, \quad t > 0, \ x \in R^1 \tag{4.1}$$

subject to the random initial condition

$$u(0, x) = \eta(x), \ x \in R^1, \tag{4.2}$$

where $a_k, k = 1, \ldots, N$ are some constants.

Let $\eta(x)$, $x \in R^1$, be the harmonizable process $\eta(x) = \int_R e^{iux} dy(u)$, with covariance function $\Gamma_\eta(x, x') = \int_R \int_R e^{i(xu-x'v)} d\Gamma_y(u, v)$.

Theorem 4.1. *Let*

$$I(t, x, \lambda) = \exp\left\{ i \left(\lambda x + t \sum_{k=1}^{N} a_k \lambda^{2k+1} (-1)^k \right) \right\}, \tag{4.3}$$

and

$$U(t, x) = \int_R I(t, x, \lambda)\, dy(\lambda). \tag{4.4}$$

If the following integrals exist

$$\int_R \lambda^s I(t, x, \lambda) dy(\lambda), \ s = 0, 1, 2, \ldots, 2N + 1, \tag{4.5}$$

and if there is a sequence $a_n > 0$, $a_n \to \infty$ as $n \to \infty$, such that for all $A > 0$ and $T > 0$ the sequence of the related integrals $\int_{-a_n}^{a_n} \lambda^s I(t, x, \lambda)\, dy(\lambda)$ converges in probability, uniformly for $|x| \le A$, $0 \le t \le T$, then $U(t, x)$ is the classical solution to the problem (4.1)–(4.2).

Proof. Since $\int_{-a_n}^{a_n} \lambda^s I(t, x, \lambda)\, dy(\lambda)$ converges in probability uniformly for $|x| \le A, 0 \le t \le T$, then there exists a subsequence $b_n > 0, b_n \to \infty$ as $n \to \infty$, such that $\int_{-b_n}^{b_n} \lambda^s I(t, x, \lambda)\, dy(\lambda)$ converges with probability one to $\int_R \lambda^s I(t, x, \lambda)$ $dy(\lambda)$, uniformly for $|x| \le A, 0 \le t \le T$. Let

$$U_{b_n}(t, x) = \int_{-b_n}^{b_n} I(t, x, \lambda)\, dy(\lambda). \tag{4.6}$$

By deriving (4.6) with respect to x and t, we easily see that

$$\sum_{k=1}^{N} a_k \frac{\partial^{2k+1} U_{b_n}(t,x)}{\partial x^{2k+1}} = \frac{\partial U_{b_n}(t,x)}{\partial t}, \qquad t > 0, x \in R^1. \tag{4.7}$$

Since

$$\frac{\partial^{2k+1} U_{b_n}(t,x)}{\partial x^{2k+1}} \quad \text{converges to} \quad \frac{\partial^{2k+1} U(t,x)}{\partial x^{2k+1}}$$

and

$$\frac{\partial U_{b_n}(t,x)}{\partial t} \quad \text{converges to} \quad \frac{\partial U(t,x)}{\partial t}$$

uniformly for $|x| \leq A$, $0 \leq t \leq T$ with probability one, we conclude that $U(t,x)$ satisfies Eq. (4.1) and $U(0,x) = \int_R e^{i\lambda x} dy(x) = \eta(x)$. □

Remark 4.1. *The integrals* $\int_R \lambda^s I(t,x,\lambda)\, dy(\lambda)$ *exist if the twofold integrals* $\int_R \int_R \lambda^s \mu^s I(t,x,\lambda)\, I(t,x,\mu)\, d\Gamma_y(\lambda,\mu)$ *exist or otherwise if*

$$\int_R \int_R |\lambda|^s\, |\mu|^s\, d\Gamma_y(\lambda,\mu) < \infty.$$

On the other side all the integrals $\int_R \lambda^s I(t,x,\lambda)\, dy(\lambda)$, $s = 0, 1, 2, \ldots, 2N+1$, exist if $\int_R \int_R |\lambda|^{2N+1}\, |\mu|^{2N+1}\, d\Gamma_y(\lambda,\mu) < \infty$.

Remark 4.2. *Under the conditions of Theorem 4.1 we can write*

$$cov\,(U(t,x), U(s,y)) = \int_R \int_R I(t,x,\lambda)\overline{I(s,y,\mu)}\, d\Gamma_y(\lambda,\mu).$$

In particular, in the case where the process $\eta(x)$ representing the initial condition is centered and stationary with a spectral function $F(\lambda)$, we have that

$$cov\,(U(t,x), U(s,y)) = \int_R I(t-s, x-y, \lambda)\, dF(\lambda)$$

and thus the solution $U(t,x)$ is stationary in space and time.

5. THE MAIN RESULT

Assumption (Ψ). Let φ be an N-function satisfying the condition Q of the Appendix; $\Psi(u) = \frac{u}{\varphi^{(-1)}(u)}$, where $\varphi^{(-1)}(u)$ is the inverse function of $\varphi(u)$. Let the function $\theta(u)$, $u > u_0$, satisfy the condition of Lemma A.2. We say that the function $\theta(u)$, $u \geq u_0 \geq 0$, satisfies the assumption Ψ if the following integral converges $\int_{0+} \Psi(\ln(\theta^{(-1)}(\varepsilon^{-1})))d\varepsilon < \infty$, where $\int_{0+} f(\varepsilon)d\varepsilon$ denotes the integral $\int_0^\delta f(\varepsilon)d\varepsilon$ for sufficiently small $\delta > 0$.

Theorem 5.1. *Let us consider the linear equation*

$$\sum_{k=1}^{N} a_k \frac{\partial^{2k+1} u(t,x)}{\partial x^{2k+1}} = \frac{\partial u(t,x)}{\partial t}, \quad t > 0, \quad x \in R^1 \tag{5.1}$$

subject to the random initial condition

$$u(0,x) = \eta(x), \quad x \in R^1. \tag{5.2}$$

Let $\eta(x)$ be the harmonizable process defined in Definition A.6, which is a strictly φ–subGaussian random process. Let $\theta(x)$, $x > u_0$ be a function satisfying the assumption Ψ. Let us assume that the following integral converges

$$\int_R \int_R |\lambda|^{2N+1} |\mu|^{2N+1} \theta(u_0 + |\lambda|^{2N+1}) \theta(u_0 + |\mu|^{2N+1}) \, d\Gamma_y(\lambda, \mu) < \infty. \tag{5.3}$$

Then $U(t,x) = \int_R I(t,x,\lambda)\, dy(\lambda)$ is the solution to the problem (5.1)–(5.2) (where $I(t,x,\lambda)$ coincides with (4.3)).

Proof. It follows from Theorem 4.1 that it is sufficient to prove that there exists a sequence $a_n > 0$, $a_n \to \infty$ as $n \to \infty$, such that the sequences $U_{n,s}(t,x) = \int_{-a_n}^{a_n} \lambda^s I(t,x,\lambda)\, dy(\lambda)$, $s = 0, 1, 2, \ldots, 2N+1$ converge uniformly in probability for $|x| \le A$, $0 \le t \le T$, where $A > 0$ and $T > 0$ are some constants. Since the random processes $U_{n,s}(t,x)$ are strictly subGaussian, then

$$\begin{aligned}
&\tau_\varphi^2(U_{n,s}(t,x) - U_{n,s}(t_1,x_1)) \\
&\quad \le C_\xi E|U_{n,s}(t,x) - U_{n,s}(t_1,x_1)|^2 \\
&\quad = C_\xi \int_{-a_n}^{a_n} \int_{-a_n}^{a_n} \lambda^s \mu^s (I(t,x,\lambda) - I(t_1,x_1,\lambda)) \\
&\qquad \times (I(t,x,\mu) - I(t_1,x_1,\mu))\, d\Gamma_y(\lambda,\mu) \\
&\quad \le C_\xi \int_R \int_R |\lambda|^s |\mu|^s |I(t,x,\lambda) - I(t_1,x_1,\lambda)| \\
&\qquad \times |I(t,x,\mu) - I(t_1,x_1,\mu)|\, |d|\, \Gamma_y(\lambda,\mu)|,
\end{aligned} \tag{5.4}$$

where C_ξ is the determining constant of the family $\{\xi(t), t \in T\}$. By assuming that the function $\theta(u)$ satisfies the assumption Ψ it is evident (bearing in mind Lemma A.2) that

$$\begin{aligned}
|I(t,x,\lambda) - I(t_1,x_1,\lambda)| = \Bigg[\Bigg(&\cos\left(\lambda x + t \sum_{k=1}^{N} a_k \lambda^{2k+1} (-1)^k \right) \\
&- \cos\left(\lambda x_1 + t_1 \sum_{k=1}^{N} a_k \lambda^{2k+1} (-1)^k \right) \Bigg)^2
\end{aligned}$$

$$+\left(\sin\left(\lambda x + t\sum_{k=1}^{N} a_k\lambda^{2k+1}(-1)^k\right) - \sin\left(\lambda x_1 + t_1\sum_{k=1}^{N} a_k\lambda^{2k+1}(-1)^k\right)\right)^2\Bigg]^{1/2}$$

$$= 2\left|\sin\frac{1}{2}\left(\lambda(x-x_1) + (t-t_1)\sum_{k=1}^{N} a_k\lambda^{2k+1}(-1)^k\right)\right|$$

$$\leq 2\left(\left|\sin\frac{x-x_1}{2}\lambda\right| + \left|\sin\frac{t-t_1}{2}\left(\sum_{k=1}^{N} a_k\lambda^{2k+1}(-1)^k\right)\right|\right)$$

$$\leq 2\left(\theta\left(u_0 + \frac{|\lambda|}{2}\right)\left(\theta\left(u_0 + \frac{1}{|x-x_1|}\right)\right)^{-1}\right.$$

$$\left. + \theta\left(u_0 + \frac{1}{2}\left|\sum_{k=1}^{N} a_k\lambda^{2k+1}(-1)^k\right|\right)\left(\theta\left(u_0 + \frac{1}{|t-t_1|}\right)\right)^{-1}\right), \tag{5.5}$$

Now it follows from (5.4) and (5.5) that

$$\sup_{\substack{|t-t_1|\leq h\\ |x-x_1|\leq h}} \tau_\varphi(U_{n,s}(t,x) - U_{n,s}(t_1,x_1)) \leq \frac{C_s}{\theta(u_0 + \frac{1}{h})}, \tag{5.6}$$

where

$$C_s^2 = 2C_\xi\int_R\int_R |\lambda|^s\,|\mu|^s\left|\theta\left(u_0 + \frac{|\lambda|}{2}\right) + \theta\left(u_0 + \frac{1}{2}\left|\sum_{k=1}^{N} a_k\lambda^{2k+1}(-1)^k\right|\right)\right|$$

$$\times\left|\theta\left(u_0 + \frac{|\mu|}{2}\right) + \theta\left(u_0 + \frac{1}{2}\left|\sum_{k=1}^{N} a_k\mu^{2k+1}(-1)^k\right|\right)\right| d\Gamma_y(\lambda,\mu).$$

It is evident that the last integrals converge since (5.3) converges.

Now the theorem follows from Theorem A.1 since

$$\sigma(h) = \frac{C_s}{\theta\left(u_0 + \frac{1}{h}\right)} \quad \text{and} \quad \sigma^{(-1)}(\varepsilon) = \frac{1}{\theta^{(-1)}\left(\frac{C_s}{\varepsilon}\right) - u_0}, \quad 0 < \varepsilon < \frac{\theta(u_0)}{C_s}$$

that is

$$\int_{0+}\Psi\left(\ln\left(\theta^{(-1)}\left(\frac{C_s}{\varepsilon}\right) - u_0\right)\right)d\varepsilon < \int_{0+}\Psi\left(\ln\left(\theta^{(-1)}\left(\frac{C_s}{\varepsilon}\right)\right)\right)d\varepsilon$$

$$= C_s\int_{0+}\Psi\left(\ln\left(\theta^{(-1)}\left(\frac{1}{\varepsilon}\right)\right)\right)d\varepsilon < \infty.$$

Corollary 5.1. *Let* $\varphi(x) = \frac{|x|^p}{p}$, $p > 1$ *for sufficiently large* x. *Then the statement of Theorem 5.1 holds if the following integral converges*

$$\int_R \int_R |\lambda\mu|^{2N+1} (\ln(1+\lambda)\ln(1+\mu))^\alpha \, d\Gamma_y(\lambda,\mu), \tag{5.1}$$

where α *is a constant such that* $\alpha > 1 - \frac{1}{p}$.

Proof. We observe that the assumption Ψ is satisfied for $\theta(x) = (\ln x)^\alpha$, where $\alpha > 1 - \frac{1}{p}$ and $x > e^\alpha$ because

$$\begin{aligned}\int_{0^+} \Psi\big(\ln\big(\theta^{(-1)}(\varepsilon^{-1})\big)\big)d\varepsilon &= \int_{0^+} \Psi\big(\varepsilon^{-1/\alpha}\big)d\varepsilon \\ &= \frac{1}{p^{1/p}} \int_{0^+} \varepsilon^{-\frac{1}{\alpha}(1-\frac{1}{p})} d\varepsilon < \infty.\end{aligned}$$

Therefore the assertion of Theorem 5.1 holds if the following integral converges

$$\int_R \int_R |\lambda\mu|^{2N+1} \left(\ln\left(e^\alpha + |\lambda|^{2N+1}\right) \ln\left(e^\alpha + |\mu|^{2N+1}\right)\right)^\alpha d\Gamma_y(\lambda,\mu) < \infty,$$

but this integral converges if the (5.7) converges.

6. STOCHASTIC INTEGRALS WITH RESPECT TO PROCESSES WITH INDEPENDENT INCREMENTS

Lemma 6.1. *Let* $\{\xi_k, k = 1, 2, \ldots\}$ *be a sequence of centered independent random variables such that* $E|\xi_k|^2 = 1$. *Let* T *be a bounded interval on* R *and let* $f_k(t)$, $k \geq 1$ *be a sequence of continuous functions on* T *such that*

$$\sum_{k=1}^{\infty} f_k^2(t) < \infty, \quad t \in T. \tag{6.1}$$

Assume that one can find a continuous function $\sigma(h)$, $h > 0$, *such that* $\sigma(h)$ *is increasing,* $\sigma(0) = 0$, *and for all sufficiently small* $\varepsilon > 0$

$$\int_0^\varepsilon \left|\ln \sigma^{(-1)}(v)\right|^{1/2} dv < \infty \tag{6.2}$$

and the following inequalities hold

$$\sup_{\substack{t,s\in T \\ |t-s|\leq h}} |f_k(t) - f_k(s)| \leq b_k \sigma(h), \tag{6.3}$$

$$\sum_{k=1}^{\infty} b_k^2 < \infty. \tag{6.4}$$

Then the series $\sum_{k=1}^{\infty} \xi_k f_k(t)$ converges uniformly for $t \in T$ with probability one.

Proof. Consider the random pseudometric, on the space T,

$$\Psi(t,s) = \left(\sum_{k=1}^{\infty} \xi_k^2 |f_k(t) - f_k(s)|^2 \right)^{1/2}.$$

Let $H_\Psi(\varepsilon) = \ln(N_\Psi(\varepsilon))$, where $N_\Psi(\varepsilon)$ is the smallest number of elements of an ε-covering of the space $(T, \Psi(t,s))$. In Theorem 3.5.5 of Ref. 5 it is proved that $\sum_{k=1}^{\infty} \xi_k f_k(t)$ converges uniformly for $t \in T$ with probability one if, with probability one, for any sufficiently small $\varepsilon > 0$,

$$\int_0^{\varepsilon} |H_\Psi(v)|^{1/2}\, dv < \infty. \tag{6.5}$$

We now prove that (6.5) holds. It follows from the assumption (6.3) that

$$\sup_{\substack{t,s \in T \\ |t-s| \le h}} \Psi(t,s) \le \left(\sum_{k=1}^{\infty} \xi_k^2 b_k^2 \right)^{1/2} \sigma(h) = \eta^{1/2} \sigma(h).$$

The series $\sum_{k=1}^{\infty} \xi_k^2 b_k^2 = \eta$ converges with probability one since $\sum_{k=1}^{\infty} E\xi_k^2 b_k^2 = \sum_{k=1}^{\infty} b_k^2 < \infty$. By consulting Ref. 5 we can see that

$$N_\Psi(u) \le \frac{|T|}{2\sigma^{(-1)}(\frac{u}{\eta})} + 1,$$

where $|T|$ is the length of T. Therefore, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
\int_0^{\varepsilon} |\ln(N_\Psi(u))|^{1/2}\, du &\le \int_0^{\varepsilon} \left| \ln\left(\frac{|T|}{2\sigma^{(-1)}(\frac{u}{\eta})} + 1 \right) \right|^{1/2} du \\
&= \int_0^{\varepsilon/\eta} \left| \ln\left(\frac{|T|}{2\sigma^{(-1)}(v)} + 1 \right) \right|^{1/2} \eta\, dv \\
&\le \eta\sqrt{2} \int_0^{\varepsilon/\eta} \left| \ln \sigma^{(-1)}(v) \right|^{1/2} dv
\end{aligned} \tag{6.6}$$

because

$$\begin{aligned}
\ln\left(\frac{T}{2\sigma^{(-1)}(v)} + 1 \right) \le \ln\left(\frac{T}{\sigma^{(-1)}(v)} \right) &\le \ln T + \left| \ln \sigma^{(-1)}(v) \right| \\
&\le 2 \left| \ln \sigma^{(-1)}(v) \right|,
\end{aligned}$$

for sufficiently small v. Therefore the integral (6.6) converges with probability one if $\int_0^\varepsilon |\ln \sigma^{(-1)}(v)|^{1/2} dv < \infty$.

Let $\xi(\lambda)$, $\lambda \in R$, be a random process with independent increments such that $E\xi(\lambda) = 0$, $E|\xi(\lambda)|^2 < \infty$ and let $F(\lambda)$ be the spectral function of this process.

Let $f(\lambda)$, $\lambda \in R$, be a function which possesses continuous derivative $f'(\lambda)$. We suppose that $\int_{-\infty}^{\infty} f(\lambda)\, d\xi(\lambda)$ exists, that is $\int_{-\infty}^{\infty} |f(\lambda)|^2\, dF(\lambda) < \infty$.

In Ref. 6 it is shown that there is a version of $\xi(\lambda)$ whose sample paths, with probability one, are measurable, bounded on any interval $[a, b]$, right continuous and have only a countable set of discontinuities. It is also assumed that the process $\xi(\lambda)$ possesses limits for $\lambda \to \pm\infty$.

In the sequel we shall consider a version of $\xi(\lambda)$, for which $\int_a^b f'(\lambda)\xi(\lambda)\, d\lambda$ exists and coincides with the Lebesgue integral. Define the following integral by means of the equality

$$\int_a^b f(\lambda)\, d\xi(\lambda) = f(b)\,\xi(b) - f(a)\,\xi(a) - \int_a^b \xi(\lambda) f'(\lambda)\, d\lambda.$$

Such integrals, in some particular cases, were introduced in Ref. 10 and in a more general situation were considered in Ref. 17.

Define $\int_{-\infty}^{\infty} f(\lambda)\, d\xi(\lambda)$ as the limit with probability one of the integrals $\int_a^b f(\lambda)\, d\xi(\lambda)$ as $a \to -\infty$, $b \to \infty$ (if this limit exists).

Theorem 6.1. *Let $g(t, \lambda)$ be a continuous function for $t \in T$, $\lambda \in R$, and let us assume also that $g'_\lambda(t, \lambda)$ exists and is continuous. Let $\xi(\lambda)$, $\lambda \in R$, be a centered random process with independent increments and spectral function $F(\lambda)$. Let the following assumptions hold*

$$\int_{-\infty}^{\infty} A^2(|\lambda|)\, dF(\lambda) < \infty, \tag{6.7}$$

where $A(\lambda) = \max_{\substack{|u|\le\lambda \\ t\in T}} |g(t, u)|$,

$$\sup_{|t-s|\le h} |g(t, \lambda) - g(s, \lambda)| \le Z(|\lambda|)\, \sigma(h), \tag{6.8}$$

where $Z(|\lambda|)$ is an increasing function such that $\int_{-\infty}^{\infty} Z^2(|\lambda|)\, dF(\lambda) < \infty$, and $\sigma(h)$, $h > 0$, is a continuous function such that $\sigma(0) = 0$ and the assumption (6.2) holds for this function.

Then the integral $\int_{-\infty}^{\infty} g(t, \lambda)\, d\xi(\lambda)$ converges uniformly for $t \in T$ with probability one.

Proof. To prove this theorem we use Lemma 6.1 and the method worked out in Ref. 10. Let us introduce the random process $y_n(u) = \xi\left(\frac{k}{n}\right)$ for $\frac{k}{n} \le u < \frac{k+1}{n}$, and consider the difference of the integrals

$$
\begin{aligned}
I_m &= \left| \int_m^{m+1} g(t,\lambda)\, d\xi(\lambda) - \int_m^{m+1} g(t,\lambda)\, dy_n(\lambda) \right| \\
&= \Big| g(t, m+1)\, \xi(m+1) - g(t, m+1)\, y_n(m+1) - g(t,m)\, \xi(m) \\
&\quad + g(t,m)\, y_n(m) - \int_m^{m+1} g'_\lambda(t,\lambda) (\xi(\lambda) - y_n(\lambda))\, d\lambda \Big| \\
&\le A(m+1)\, |\xi(m+1) - y_n(m+1)| + A(m)\, |\xi(m) - y_n(m)| \\
&\quad + B(m) \int_m^{m+1} |\xi(\lambda) - y_n(\lambda)|\, d\lambda,
\end{aligned}
$$

where $B(m) = \max_{\substack{t\in T \\ m\le\lambda\le m+1}} |g'_\lambda(t,\lambda)|$. The properties of the process $\xi(\lambda)$ guarantee that $I_m \to 0$ as $m \to \infty$ uniformly for $t \in T$ with probability one.

For any $\varepsilon > 0$, there exists a number $n_{\varepsilon,m}$ such that, with probability larger than $1 - \frac{\varepsilon}{2^{|m|+2}}$, the following inequality holds

$$
\left| \int_m^{m+1} g(t,\lambda)\, d\xi(\lambda) - \int_m^{m+1} g(t,\lambda)\, dy_{n_{\varepsilon,m}}(\lambda) \right| < \frac{\varepsilon}{2^{|m|+2}}. \tag{6.9}
$$

Consider now the random process $y_\varepsilon(\lambda) = y_{n_{\varepsilon,m}}(\lambda)$ as $m \le \lambda \le m+1$. For $A_1 < A_2$ the following inequality holds

$$
\begin{aligned}
\left| \int_{A_1}^{A_2} g(t,\lambda)\, d\xi(\lambda) \right| &\le \Big| \int_{A_1}^{A_2} g(t,\lambda)\, d\xi(\lambda) \\
&\quad - \int_{A_1}^{A_2} g(t,\lambda)\, dy_{n_{\varepsilon,m}}(\lambda) \Big| + \left| \int_{A_1}^{A_2} g(t,\lambda)\, dy_{n_{\varepsilon,m}}(\lambda) \right|.
\end{aligned} \tag{6.10}
$$

It follows from (6.9) that $| \int_{A_1}^{A_2} g(t,\lambda) d\xi(\lambda) - \int_{A_1}^{A_2} g(t,\lambda) dy_{n_{\varepsilon,m}}(\lambda)| \le \varepsilon$ with probability larger than $1-\varepsilon$. Therefore there exists a sequence $\varepsilon_k \to 0$ as $k \to \infty$, such that with probability one uniformly for all $A_1, A_2, t \in T$

$$
\int_{A_1}^{A_2} g(t,\lambda)\, dy_{n,\varepsilon_k}(\lambda) \to \int_{A_1}^{A_2} g(t,\lambda)\, d\xi(\lambda). \tag{6.11}
$$

Therefore the assertion of the theorem holds true if the integral $\int_{-\infty}^{\infty} g(t,\lambda)\, dy_{n,\varepsilon}(\lambda)$ converges uniformly as $t \in T$ for any $\varepsilon > 0$ with probability one (see inequality (6.10)). Note that, for $\lambda_{s+1} > \lambda_s$,

$$
I(t) = \int_{-\infty}^{\infty} g(t,\lambda)\, dy_{n,\varepsilon}(\lambda) = \sum_{s=-\infty}^{\infty} g(t,\lambda_s) (\xi(\lambda_{s+1}) - \xi(\lambda_s))
$$

$$= \sum_{s=-\infty}^{\infty} g(t, \lambda_s)\, \delta_s 1(\delta_s \neq 0) \frac{\xi(\lambda_{s+1}) - \xi(\lambda_s)}{\delta_s}$$

$$= \sum_{s=-\infty}^{\infty} g(t, \lambda_s)\, \delta_s 1(\delta_s \neq 0)\, \eta_s,$$

where $\delta_s^2 = F(\lambda_{s+1}) - F(\lambda_s)$ and η_s are independent random variables such that $E\,|\eta_s|^2 = 1$. We check that the assumptions of Lemma 6.1 hold true for the series $I(t)$. It follows from the assumption (6.7) that

$$\sum_{s=-\infty}^{\infty} g^2(t, \lambda_s)\, \delta_s^2 1^2(\delta_s \neq 0)$$

$$\leq \sum_{s=-\infty}^{\infty} A^2(|\lambda_s|)(F(\lambda_{s+1}) - F(\lambda_s)) \leq \int_{-\infty}^{\infty} A^2(\lambda)\, dF(\lambda) < \infty.$$

We now check that the assumptions (6.3) and (6.4) hold. Indeed, it follows from assumption (6.8) that $\sup_{\substack{t,u\in T\\ |t-u|\leq h}} |g(t, \lambda_s) - g(u, \lambda_s)|\, \delta_s \leq Z(|\lambda_s|)\, \delta_s \sigma(|h|)$ and

$$\sum_{s=-\infty}^{\infty} Z^2(|\lambda_s|)\, \delta_s^2 = \sum_{s=-\infty}^{\infty} Z^2(|\lambda_s|)(F(\lambda_{s+1}) - F(\lambda_s))$$

$$\leq \int_{-\infty}^{\infty} Z^2(|\lambda|)\, dF(\lambda) < \infty.$$

□

Theorem 6.2. *Consider the linear Eq. (4.1)*

$$\sum_{k=1}^{N} a_k \frac{\partial^{2k+1} u(t, x)}{\partial x^{2k+1}} = \frac{\partial u(t, x)}{\partial t}, \quad t > 0,\ x \in R^1$$

subject to the random initial condition $u(0, x) = \eta(x), x \in R^1$, where $\eta(x) = \int_{-\infty}^{\infty} e^{i\lambda x} d\xi(\lambda)$, $\xi(\lambda)$ is a random process with independent increments and spectral function $F(\lambda)$. Let $\theta(x)$, $x > x_0$, be a function satisfying the conditions of Lemma A.2 and such that for sufficiently small $\varepsilon > 0$ $\int_0^{\varepsilon} \ln \theta^{(-1)}(u^{-1}) du < \infty$.

Assume that the following integral converges $\int_R |\lambda|^{4N+2}\, \theta^2(u_0 + |\lambda|^{2N+1})\, dF(\lambda) < \infty$.

Then $U(t, x) = \int_R I(t, x, \lambda)\, dy(\lambda)$, where

$$I(t, x, \lambda) = \exp\left\{ i \left(\lambda x + t \sum_{k=1}^{N} a_k \lambda^{2k+1} (-1)^k \right) \right\},$$

is the classical solution of problem (4.1)–(4.2).

Proof. The proof is similar to that of Theorem 5.1 with Theorem A.1 replaced by Theorem 6.1.

7. NOTE ON GENERALIZED SOLUTIONS

Generalized solutions for Eq. (4.1), with the random initial data (4.2) and $\eta(x) = \int_R e^{iux}\, dy(u)$, are given by processes of the form

$$U(t, x) = \int_R I(t, x, \lambda)\, dy\,(\lambda). \tag{7.1}$$

where $I(t, x, \lambda) = \exp\{i(\lambda x + t \sum_{k=1}^{N} a_k \lambda^{2k+1}(-1)^k)\}$, provided that the integral (7.1) converges uniformly in probability for $|x| \le A$, $0 < t \le T$ for all A, T. The condition under which the integral (7.1) converges is given below.

Condition G. There exists a sequence $a_n > 0$, $a_n \to \infty$ as $n \to \infty$, such that for all $A > 0$ and $T > 0$ the sequence of the integrals $\int_{-a_n}^{a_n} I(t, x, \lambda)\, dy\,(\lambda)$ converges in probability to (7.1), uniformly for $|x| \le A$, $0 \le t \le T$.

Condition **G** implies that there exists a subsequence $a_{n_k} > 0$ of the sequence a_n such that $\int_{-a_{n_k}}^{a_{n_k}} I(t, x, \lambda) dy(\lambda)$ converges almost surely to (6.1), uniformly for $|x| \le A$, $0 \le t \le T$.

By analyzing the proofs of the results of Secs. 4, 5 and 6 we arrive at the following statements.

Let $\eta(x)$ be a harmonizable process which is strictly φ–subGaussian and the function $\theta\,(x)\,, x > u_0$, be a function satisfying the assumption Ψ. Then condition **G** holds if the following integral converges

$$\int_R \int_R \theta(u_0 + |\lambda|^{2N+1})\, \theta(u_0 + |\mu|^{2N+1})\, d\Gamma_y(\lambda, \mu) < \infty. \tag{7.2}$$

When $\eta(x)$ is a strictly φ–subGaussian stationary process $\eta(x) = \int_R e^{iux} d\xi(u)$, where $\xi(u)$ is a centered process with uncorrelated increments $(E\eta(x+\tau)\,\overline{\eta(x)} = \int_R e^{i\tau\lambda} dF(\lambda))$, the condition (7.2) becomes

$$\int_R \theta^2(u_0 + |\lambda|^{2N+1})\, dF(\lambda) < \infty. \tag{7.3}$$

Let $\theta(x)$, $x > x_0$, be a function satisfying the conditions of Lemma A.2 such that, for sufficiently small $\varepsilon > 0$, $\int_0^\varepsilon \ln \theta^{(-1)}(u^{-1})\, du < \infty$. Then condition **G** holds if $\int_R \theta^2(u_0 + |\lambda|^{2N+1})\, dF\,(\lambda) < \infty$.

8. APPENDIX

Definition A.1. [(5)] Let $\varphi = \{\varphi(x), x \in R\}$ be a continuous even convex function. The function φ is an Orlicz $N-$function if $\varphi(0) = 0$, $\varphi(x) > 0$ as $x \neq 0$ and the following conditions hold: $\lim_{x\to 0} \frac{\varphi(x)}{x} = 0$, $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$.

Definition A.2. [(5)] Let $\varphi = \{\varphi(x), x \in R\}$ be an $N-$function. The function φ^* defined by $\varphi^*(x) = \sup_{y\in R}(xy - \varphi(y))$ is called the Young-Fenchel transform of φ.

Remark A.1. [(5)] *The Young-Fenchel transform of an $N-$function is again an $N-$function and the following inequality holds (Young-Fenchel inequality) $xy \leq \varphi(x) + \varphi^*(y)$ as $x > 0$, $y > 0$.*

Condition Q. Let φ be an $N-$function which satisfies $\liminf_{x\to 0} \frac{\varphi(x)}{x^2} = C > 0$.The case $C = \infty$ is possible.

Definition A.3. [(8)] Let φ be an $N-$function satisfying Condition Q and $\{\Omega, B, P\}$ be a standard probability space. The random variable ξ belongs to the space $Sub_\varphi(\Omega)$, if $E\xi = 0$, $E\exp\{\lambda\xi\}$ exists for all $\lambda \in R$ and there exists a constant $a > 0$ such that the following inequality holds for all $\lambda \in R$

$$E\exp\{\lambda\xi\} \leq \exp\{\varphi(\lambda a)\}. \tag{8.1}$$

The space $Sub_\varphi(\Omega)$ is a Banach space with respect to the norm ([18])

$$\tau_\varphi(\xi) = \sup_{\lambda\neq 0} \frac{\varphi^{(-1)}(\ln E\exp\{\lambda\xi\})}{|\lambda|}.$$

Examples of φ–subGaussian random variables can be found in the paper[8] and in the book[5].

Definition A.4. [(20)] A family Δ of random variables $\xi \in Sub_\varphi(\Omega)$ is called strictly φ–subGaussian if there exists a constant C_Δ such that for all finite sets of random variables $\xi_i \in \Delta$ the following inequality holds

$$\tau_\varphi\left(\sum_{i\in I}\lambda_i\xi_i\right) \leq C_\Delta \left|E\left(\sum_{i\in I}\lambda_i\xi_i\right)^2\right|^{1/2}. \tag{8.2}$$

The constant C_Δ is called the determining constant of the family Δ.

Lemma A.1. [(20)] *The linear closure of a strictly φ–subGaussian family Δ in the space $L_2(\Omega)$ is the strictly φ–subGaussian family with the same determining constant.*

Definition A.5. The random process $\xi = \{\xi(t), t \in T\}$ is called (strictly) φ–subGaussian if all random variables $\xi(t), t \in T$, are (strictly) φ–subGaussian and $\sup_{t\in T} \tau_\varphi(\xi(t)) < \infty$.

Definition A.6. A harmonizable random process $\eta(x) = \int_R e^{iux} dy(u)$ is a strictly φ–subGaussian harmonizable random process if the process y is strictly φ–subGaussian.

Let (T, d) be a compact metric space and $C(T)$ is the Banach space of continuous functions with uniform norm. Let $X_k = \{X_k(t), t \in T\}$ be a sequence of φ–subGaussian random processes such that $X_k \in C(T)$. The general conditions of convergence in probability of X_k in the space $C(T)$ are presented in the book (5). In the paper(20) these conditions are presented for the case where T is a finite-dimensional space.

Theorem A.1. [(20)] *Let R^k be a k-dimensional space, $d(t,s) = \max_{1\le i\le k} |t_i - s_i|$, $T = \{0 \le t_i \le T_i,\ i = 1, 2, \ldots, k\}$, $T_i > 0$; $X_n = \{X_n(t), t \in T\}$ be a sequence of φ–subGaussian random processes such that $X_n \in C(T)$. Let us assume also that there exists a continuous increasing function $\sigma = \{\sigma(h), h > 0\}$, $\sigma(h) \to 0$ as $h \to 0$, such that*

$$\sup_{d(t,s)\le h} \tau_\varphi(X_n(t) - X_n(s)) \le \sigma(h) \tag{8.3}$$

and

$$\int_{0+} \Psi\left(\ln \frac{1}{\sigma^{(-1)}(\varepsilon)}\right) d\varepsilon < \infty, \tag{8.4}$$

where $\Psi(u) = \frac{u}{\varphi^{(-1)}(u)}$, $\sigma^{(-1)}(u)$ is the inverse function of $\sigma(u)$, $\varphi^{(-1)}(u)$ is the inverse function of $\varphi(u)$, for $u > 0$, and $\int_{0+} f(\varepsilon)d\varepsilon$ denotes $\int_0^\delta f(\varepsilon)d\varepsilon$ for sufficiently small $\delta > 0$. If the sequence of processes $X_n(t)$, $n \ge 1$, converges in probability to $X(t)$ for all $t \in T$, then $X_n(t)$ converges in probability to $X(t)$ in the space $C(T)$.

Lemma A.2. [(19)] *Let $\theta(u)$, $u \ge u_0 \ge 0$, be a continuous, increasing function such that $\theta(u) > 0$ and the function $\frac{u}{\theta(u)}$ is non-decreasing for $u > u_0$, where $u_0 \ge 0$ is a constant. Then for all $u, v \ne 0$*

$$\left|\sin \frac{u}{v}\right| \le \frac{\theta(|u| + u_0)}{\theta(|v| + u_0)}. \tag{8.5}$$

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